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Gabriela Stangenhaus and H. T. David*

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A ROBUST CRAMÉR-RAO ANALOGUE

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1. INTRODUCTION AND SUMMARY

We give an analogue of the usual uni-parameter Cramér-Rao development, in which unbiasedness is replaced by median-unbiasedness, variance is replaced by a dispersion measure first proposed by Alamo (1964), which we call local kurtosis, and the information in the sample is computed in terms of the first absolute moment of the sample score, rather than the second moment.

Given a sample $\underline{x}$, with density $f(\underline{x};\theta)$, consider statistics that are median-unbiased for θ; i.e., statistics δ such that

$$\int_{\underline{x}:\delta(\underline{x}) \leq \theta} f(\underline{x};\theta)d\underline{x} = \int_{\underline{x}:\delta(\underline{x}) \geq \theta} f(\underline{x};\theta)d\underline{x} = \tfrac{1}{2} . \qquad (1.1)$$

Given also the density $g_\delta(\cdot;\theta)$ of δ, define $K(\delta;\theta)$, the local kurtosis of δ, by

$$K(\delta;\theta) = (2g_\delta(0;\theta))^{-1}. \qquad (1.2)$$

We show below that, in the "regular" case, (1.1) implies the (L_1 Cramér-Rao) inequality

$$K(\delta;\theta) \geq (I_1(\theta))^{-1}, \qquad (1.3)$$

where I_1 is an L_1 analogue of Fisher's information:

$$I_1(\theta) = \int \left| \frac{\delta}{\delta\theta} \ln f(\underline{x};\theta)\right| f(\underline{x};\theta)d\underline{x}, \qquad (1.4)$$

(1.3) being equally valid when the derivative in (1.4) is only a left or a right derivative.

A "not-necessarily-regular" version of (1.3); i.e., a Chapman-Robbins analogue (1951, relation (5)), is:

$$K(\delta;\theta) \geq \left(\lim_{|h|\to 0} \int \left|\frac{f(\underline{x};\theta+h) - f(\underline{x};\theta)}{hf(\underline{x};\theta)}\right| f(\underline{x};\theta)d\underline{x}\right)^{-1} . \qquad (1.5)$$

A necessary and sufficient condition for equality in (1.3) is:

$$\int_{\underline{x}:\delta(\underline{x}) \leq \theta} \left|\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta)\right| f(\underline{x};\theta)d\underline{x} = \left|\int_{\underline{x}:\delta(\underline{x}) \leq \theta} \left(\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta)\right) f(\underline{x};\theta)d\underline{x}\right| \qquad (1.6a)$$

and

$$\int_{\underline{x}:\delta(\underline{x}) \geq \theta} \left|\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta)\right| f(\underline{x};\theta)d\underline{x} = \left|\int_{\underline{x}:\delta(\underline{x}) \geq \theta} \left(\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta)\right) f(\underline{x};\theta)d\underline{x}\right|, \qquad (1.6b)$$

and a family of pairs (f,δ) meeting (1.1) and (1.6) consists of the densities

$$f_L(x;\theta) = C \exp L(x-\theta), \quad -\infty < x < \infty , \qquad (1.7)$$

where $\Theta = R$ and $L(\cdot)$ is symmetric, satisfying certain smoothness conditions (cf. section 4), the associated statistics δ being the maximum-likelihood estimates $\delta_L(\underline{x})$:

$$\sum_i L(x_i - \delta_L(\underline{x})) = \sup_\theta \sum_i L(x_i - \theta). \qquad (1.8)$$

$K(\delta;\theta)$ in (1.2) is a local and scalar version of the "risk curve" $a(\cdot,\theta,\delta)$, as defined by relation (1.1) in Birnbaum (1961). The curve $a(\cdot,\theta,\delta)$, monotone increasing to the left of θ and monotone de-

creasing to the right, measures tail size of the distribution of δ with respect to θ, and the dispersion of δ about θ is measured in non-scalar qualitative fashion by the elevation of this curve. When δ is median-unbiased and possesses a density $g_\delta(\cdot;\theta)$, then $a(0,\theta,\delta) = \frac{1}{2}$ and $a'(\theta-,\theta,\delta) = |a'(\theta+,\theta,\delta)| = g_\delta(\theta,\theta)$, so that $K(\delta;\theta)$ is indeed a natural scalar summarization of the elevation of $a(\cdot,\theta,\delta)$.

Alamo (1964) proposed $K(\delta;\theta)$ in connection with a Cramér-Rao analogue for median-unbiased estimation, but failed to replace the Fisherian bound based on $I_2(\theta)$ with the sharper one based on $I_1(\theta)$, and thus did not achieve the full potential of his invention. Relations (1.3) and (1.5) are given in Stangenhaus and David (1979), but under restrictive regularity assumptions arising from deriving (1.3) and (1.5) as limits of analogous L_p relations, $p > 1$. We do owe our use of the term "local kurtosis" to the development in that paper, in that ordinary kurtosis does turn out, essentially, to be the natural measure of dispersion for the case $p = 4$.

Modifications I_r, $r > 1$, of Fisher's information I_2 appear for example in Barankin's study of minimum s-norm unbiased estimation, $1/s + 1/r = 1$, while I_1 itself appears not to have been suggested heretofore. It must be noted, however, that all departures I_r, $r \neq 2$, from Fisher's I_2 may in fact be problematical qua information. None for example is, as required by Schutzenberger's (1951) axioms for information measures, the expectation of a linear functional of $\ln f$.

Though the present point of view appears to be somewhat distinct, our conclusions (or at least some of their specializations) do accord with historical precedent. With regard to our Example 2.1 for example, Birnbaum (1961) has identified the sample mid-range as "admissible"

median-unbiased in the case of samples of size two from the uniform
distribution. Again, with regard to our Example 4.1, Laplace is
credited in Birnbaum (1964) with having identified the sample mean
as minimum-expected-absolute-deviation among median-unbiased estimates,
in the case of normal samples.

Each facet of our development, be it the median-unbiasedness or
local kurtosis of δ, or the information I_1, exhibits its own type
of insensitivity to tail behavior, the tail in question, in the case
of I_1, being that of the distribution of the sample score $\frac{\delta}{\delta\theta} \ln f(\underline{X};\theta)$.
Thus our presentation seems properly viewed in the light of the broad
area of robust estimation (Huber, 1981).

2. A LOWER BOUND FOR THE LOCAL KURTOSIS OF MEDIAN-UNBIASED ESTIMATES

We consider an n-dimensional sample $\underline{X} = (X_1,\ldots,X_n)$ with a
density $f(\underline{x};\theta)$ over R^n, θ belonging to an open interval O of
R, and a statistic $\delta(\underline{X}) \overset{\Delta}{=} Y$ such that, for $\theta \epsilon O$,

$$Y \text{ possesses a density } g(y;\theta) \text{ over } R . \tag{2.1}$$

For such δ one has, for θ', $\theta'' \epsilon O$,

$$\underline{\gamma}(\theta',\theta'') \overset{\Delta}{=} \int_{-\infty}^{\theta'} g_\delta(y;\theta'')dy = \int_{\underline{x}:\delta(\underline{x}) \le \theta'} f(\underline{x};\theta'')d\underline{x} \overset{\Delta}{=} \underline{\phi}(\theta',\theta''), \tag{2.2}$$

and analogously for the corresponding "upper-tail" functions $\overline{\gamma}(\theta',\theta'')$
and $\overline{\phi}(\theta',\theta'')$.

When δ is median-unbiased for θ, one has as well (cf. (1.1)),
for $\theta \epsilon O$,

$$\underline{\phi}(\theta,\theta) = \underline{\gamma}(\theta,\theta) = \overline{\gamma}(\theta,\theta) = \overline{\phi}(\theta,\theta) = \tfrac{1}{2} , \tag{2.3}$$

and, for θ and positive h such that both θ and $\theta + h$ are in $\mathcal{O}$, (2.2) and (2.3) imply that

$$\int_\theta^{\theta+h} g_\delta(y;\theta+h)dy \overset{\Delta}{=} \underline{\gamma}(\theta+h,\theta+h) - \underline{\gamma}(0,\theta+h)$$

$$= \underline{\phi}(\theta+h,\theta+h) - \underline{\phi}(\theta,\theta+h) = \underline{\phi}(\theta,0) - \underline{\phi}(0,\theta+h) \tag{2.4}$$

$$\overset{\Delta}{=} \int\limits_{\underline{x}:\delta(\underline{x}) \le \theta} [f(\underline{x};\theta) - f(\underline{x};\theta+h)]d\underline{x} ,$$

where the first, second and fourth equalities are due to (2.2), and the third to (2.3).

Suppose now in addition that, for $\theta \in \mathcal{O}$, $g_\delta(y;\theta)$ is continuous in y, in a neighborhood

$$N_\theta \quad \text{of} \quad y = \theta, \ N_\theta \in \mathcal{O} \tag{2.5}$$

Then for positive h such that $\theta + h \in N_\theta$, the law of the mean gives, for $\theta \in \mathcal{O}$,

$$\int_\theta^{\theta+h} g_\delta(y;\theta+h)dy = h \cdot g_\delta(\theta+h \cdot \lambda_h; \ \theta+h) \tag{2.6}$$

for the LHS of (2.4), where $0 \le \lambda_h \le 1$, and, dividing (2.4) by h and taking absolute values, one finds

$$g(\theta+h \cdot \lambda_h; \ \theta+h) \le \int\limits_{\underline{x}:\delta(\underline{x}) \le \theta} |f(\underline{x};\theta+h) - f(\underline{x};\theta)/h|d\underline{x} . \tag{2.7}$$

and (2.7) is seen to be equally valid for negative h such that $\theta + h \in N_\theta$.

An analogous argument using $\overline{\gamma}$ and $\overline{\phi}$ leads as well to

$$g(\theta+h\cdot\lambda_h; \ \theta+h) \leq \int |f(\underline{x};\theta+h) - f(\underline{x};\theta)|d\underline{x} \qquad (2.8)$$
$$\underline{x}:\delta(\underline{x}) \geq 0$$

and, adding (2.8) to (2.7), one concludes that

$$\lim_{|h| \to 0} 2g(\theta+h\cdot\lambda_h; \ \theta+h) \leq \lim_{|h| \to 0} \int |f(\underline{x};\theta+h) - f(\underline{x};\theta)/h|d\underline{x}. \quad (2.9)$$

When in addition, for $\theta \epsilon 0$,

$$g_\delta(\cdot;\cdot) \quad \text{is continuous at} \quad (\theta,\theta), \qquad (2.10)$$

relation (2.9) yields relation (1.5).

We shall say that a pair (f,δ) satisfying regularity conditions (2.1), (2.5) and (2.10), and also (1.1) plus (1.5) with equality, is L_1-optimal.

Example 2.1. Consider $f = f_u(x;0)$, equal to unity for $0 - \frac{1}{2} \leq x \leq \theta + \frac{1}{2}$, and to zero otherwise, and also δ the sample mid-range (smr). The pair (f_u, smr) is L_1-optimal:

To begin with (David, 1981; problem 2.3.5(b)),

$$g_{smr}(y;0) = n(1-2|y-\theta|)^{n-1}, \ \theta - \frac{1}{2} \leq y \leq 0 + \frac{1}{2},$$

so that (f_u, smr) satisfies (1.1), (2.1), (2.5) and (2.10), and also

$$K(\text{smr};\theta) = (2g_{smr}(0;\theta))^{-1} = 1/2n .$$

It therefore remains to show that the RHS of (1.5) equals $1/2n$. But, writing the integrand in the RHS of (1.5) as $|h|^{-1}|f(\underline{x};\theta+h) - f(\underline{x};0)|$, we see that the integral in the RHS of (1.5) is the content of the symmetric difference of two unit n-cubes displaced with respect to each other by an amount $\sqrt{n}\, h$ along the equiangular line. This content is $2n\, h$ to order h, so that the RHS of (1.5) is indeed $1/2n$.

3. THE REGULAR CASE

Consistently with precedent, we identify the regular case as that for which, when $\theta \in \Theta$,

$$\left|\frac{\delta}{\delta\theta'} \ln f(\underline{x};\theta')\right| \leq G_\theta(\underline{x}); \quad \theta' \in N_\theta, \quad \underline{x} \in R^n, \tag{3.1a}$$

where

$$\int G_\theta(\underline{x}) f(\underline{x};0) d\underline{x} < +\infty, \quad \theta \in \Theta, \tag{3.1b}$$

and where the derivative may be construed as one-sided, say as a right derivative. In that case the RHS of (1.5) is bounded below by

$$\left(\lim_{h\to 0+} \int \left|\frac{f(\underline{x};\theta+h) - f(\underline{x};\theta)}{hf(\underline{x};0)}\right| f(\underline{x};0) d\underline{x}\right)^{-1} =$$

$$\left(\int \left|\lim_{h\to 0+} \left(\frac{f(\underline{x};\theta+h) - f(\underline{x};\theta)}{hf(\underline{x};\theta)}\right)\right| f(\underline{x};\theta) d\underline{x}\right)^{-1} = (I_1(\theta))^{-1},$$

and relation (1.5) is seen to imply relation (1.3) under (3.1).

To study achievability for (1.3), we write it in the form

$$K(\delta;\theta) \geq \left(\int_{\underline{x}:\delta(\underline{x}) \geq 0} \left|\frac{\delta}{\delta\theta} \ln f(\underline{x};0)\right| f(\underline{x};0) d\underline{x} + \int_{\underline{x}:\delta(\underline{x}) \leq 0} \left|\frac{\delta}{\delta\theta} \ln f(\underline{x};0)\right| f(\underline{x};0) d\underline{x}\right)^{-1}. \tag{3.2}$$

Now divide (2.4) by h, as was done in obtaining (2.7), and let h tend to zero under assumption (3.1), but without absolute values taken. We find

$$g_\delta(0,\theta) = \left|\int_{\underline{x}:\delta(\underline{x}) \leq 0} \left(\frac{\delta}{\delta\theta} \ln f(\underline{x};0)\right) f(\underline{x};0) d\underline{x}\right|,$$

and, correspondingly,

$$g_\delta(\theta,\theta) = \left|\int(\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta))f(\underline{x};\theta)d\underline{x}\right|;$$
$$\underline{x}:\delta(\underline{x}) \geq \theta$$

in other words,

$$2g_\delta(\theta,\theta) = \left|\int(\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta))f(\underline{x};\theta)d\underline{x}\right|$$
$$\underline{x}:\delta(\underline{x}) \leq \theta$$

$$+ \left|\int(\frac{\delta}{\delta\theta} \ln f(\underline{x};0))f(\underline{x};0)d\underline{x}\right|. \tag{3.3}$$
$$\underline{x}:\delta(\underline{x}) \geq 0$$

Relations (3.2) and (3.3) make clear that, indeed, conditions (1.6) are necessary and sufficient for equality in (1.3), given conditions (2.1), (2.5), (2.10) and (3.1).

We shall say that a pair (f,δ) satisfying regularity conditions (2.1), (2.5), (2.10) and (3.1), and also (1.1) and (1.6), is regular L_1-optimal.

<u>Example 3.1.</u> Consider $f = f_\mathcal{L}(x;\theta) = \frac{1}{2} \exp -|x-\theta|$, $-\infty < x < +\infty$, and also δ the median (med) of a sample of odd size n . The pair $(f_\mathcal{L}$, med) is regular L_1-optimal:

To begin with, it is seen by inspection of $g_{med}(y;\theta)$ (Johnson and Kotz, 1970; p. 25) that $(f_\mathcal{L}$, med) satisfies (1.1), (2.1), (2.5) and (2.10). In addition,

$$\frac{\delta}{\delta\theta} \ln f_\mathcal{L}(\underline{x};\theta) = [\# \text{ of } x_i\text{'s} > \theta] - [\# \text{ of } x_i\text{'s} \leq \theta], \tag{3.4}$$

and (3.4) is of one sign both in the region med $\leq \theta$ and in the region med > 0, so that (1.6) is true. Moreover (3.4) implies that $\left|\frac{\delta}{\delta\theta} \ln f_\mathcal{L}(\underline{x};\theta)\right| \leq n$, so that (3.1) is seen to hold with $G_0(\underline{x}) = n$.

4. AN L_1 EXPONENTIAL FAMILY

As indicated in section 1, the pair (f_L, δ_L) in (1.7) and (1.8) may be shown to be regular L_1-optimal under certain smoothness conditions on the symmetric function $L(\cdot)$. To begin with, if (1.8) is to define δ_L uniquely, it must be that

$$L(\cdot) \text{ is strictly concave on } R. \tag{4.1}$$

Next, if (1.8) is to reduce to the analytically tractable condition

$$\Sigma L'(x_i - \delta_L(\underline{x})) = 0, \tag{4.2}$$

it must be that

$$L(\cdot) \text{ possesses a derivative } L'(\cdot) \text{ on } R. \tag{4.3}$$

Conditions (4.1) and (4.3) together of course also imply that

$$\Sigma L'(x_i - \theta) \text{ is strictly increasing in } \theta, \tag{4.4}$$

and are in fact sufficient to insure (1.1) and (1.6), as is now demonstrated in Lemmas (4.2) and (4.3).

Lemma 4.1. Under conditions (4.1) and (4.3), the distribution of δ_L is symmetric about 0 .

Proof. Under the stated conditions and their implications (4.2) and (4.4), $\Pr\{\delta_L(\underline{X}) < \theta + \varepsilon\}$

$\quad = \Pr\{\Sigma L'(X_i - \theta - \varepsilon) > 0\} = \Pr\{-\Sigma L'(X_i - \theta - \varepsilon) < 0\}$

$\quad = \Pr\{\Sigma L'(-(X_i - \theta) + \varepsilon) < 0\} = \Pr\{\Sigma L'(X_i - \theta + \varepsilon) < 0\}$

$\quad = \Pr\{\Sigma L'(X_i - (\theta - \varepsilon)) < 0\} = \Pr\{\delta_L(\underline{X}) > \theta - \varepsilon\}$.

Lemma 4.2. Under conditions (4.1) and (4.3), δ_L is median-unbiased for θ (i.e., (1.1) holds).

Proof. By Lemma 4.1.

Lemma 4.3. Under conditions (4.1) and (4.3), (f_L, δ_L) satisfies (1.6).

Proof. Since

$$\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta) = - \Sigma L'(x_i - \theta), \tag{4.5}$$

we find, analogously to the argument in the proof of Lemma 4.1, that,
under conditions (4.1) and (4.3), and their implications (4.2) and
(4.4),

$$\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta) \gtreqless 0 \leftrightarrow \delta_L(\underline{x}) \gtreqless 0.$$

Hence $\frac{\delta}{\delta\theta} \ln f(\underline{x};\theta)$ is of one sign both when $\delta_L(\underline{x}) \geq 0$ and when
$\delta_L(\underline{x}) \leq 0$, which establishes (1.6).

Verifying condition (3.1) requires one further assumption; namely,
that there exist $K(\varepsilon)$ such that, for $h \in (-\varepsilon, \varepsilon)$ and $x \in R$,

$$|L'(x-h)| \leq K(\varepsilon) \cdot |L'(x)|. \tag{4.6}$$

Lemma 4.4. Condition (3.1) is met under (4.1), (4.3) and (4.6).

Proof. Given assumption (4.6), for $\theta' = \theta + h$, $h \in (-\varepsilon, \varepsilon)$, and $x \in R$,

$$|L'(x-\theta')| = |L'((x-\theta) - h)|$$

$$\leq K(\varepsilon) \cdot |L'(x-\theta)| \triangleq g_{\theta,\varepsilon}(x), \tag{4.7}$$

and

$$\int g_{\theta,\varepsilon}(x) f_L(x;\theta)dx =$$

$$C \cdot K(\varepsilon) [\int_\theta^\infty |L'(x-\theta)| e^{L(x-\theta)}dx + \int_{-\infty}^\theta |L'(x-\theta)| e^{L(x-\theta)}dx] \tag{4.8}$$

$$= C \cdot K(\epsilon) [-\int_0^\infty L'(z) e^{L(z)} dz + \int_{-\infty}^0 L'(z) e^{L(z)} dz]$$

$$= 2C \, e^{L(0)} K(\epsilon).$$

It follows that the function

$$G_{\theta,\epsilon}(\underline{x}) = \Sigma g_{\theta,\epsilon}(x_i)$$

satisfies (3.1), since, by (4.7), for $\theta' \epsilon (\theta-\epsilon, \theta+\epsilon)$,

$$\left| \frac{\delta}{\delta \theta'} \ln f_L(\underline{x};\theta') \right| \le \Sigma |L'(x_i-\theta')|$$

$$\le \Sigma g_{\theta,\epsilon}(x_i) \overset{\Delta}{=} G_{\theta,\epsilon}(\underline{x}),$$

and also, by (4.8),

$$\int G_{\theta,\epsilon}(\underline{x}) \, f_L(\underline{x};\theta) d\underline{x} = \Sigma \int g_{\theta,\epsilon}(x_i) \, f_L(x_i;\theta)$$

$$= 2n \, C \, e^{L(0)} K(\epsilon) \ .$$

It remains to verify (2.1), (2.5) and (2.10). Unfortunately it seems difficult to adduce tractable conditions on $L(\cdot)$ insuring (2.1) that are not at the same time unreasonably restrictive. Thus we simply add (2.1) itself to the list of requirements for $L(\cdot)$:

$$L(\cdot) \text{ is such that } \delta_L \text{ possesses a density } g_{\delta_L} . \qquad (4.9)$$

Given (4.9), it is however possible to condense (2.5) and (2.10) into a single condition, through the observation that θ is a location parameter for the distribution of δ_L . This last is seen by adding and subtracting θ inside the main brackets of (4.2), whereupon it follows that $\delta_L(\underline{X}) - \theta$ is a function only of the parameter-free

quantities $(X_1-\theta)$. At any rate the density g_{δ_L} of (4.9) may now be written

$$g_{\delta_L}(y;\theta) = g(y-\theta), \tag{4.10}$$

so that conditions (2.5) and (2.10) are seen to reduce to the single condition

$$g \text{ in (4.10) is continuous in a neighborhood of 0.} \tag{4.11}$$

In summary, then, conditions on $L(\cdot)$ insuring that the pair (f_L, δ_L) in (1.7) and (1.8) is regular L_1-optimal are (4.1), (4.3), (4.6), (4.9) and (4.11).

Example 4.1. $(N(\theta,1), \overline{X})$ is regular L_1-optimal.

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